

A decomposable branching process in a Markovian environment

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Abstract

A population has two types of individuals, each occupying an island. One of those, where individuals of type 1 live, offers a variable environment. Type 2 individuals dwell on the other island, in a constant environment. Only one-way migration ($1 \rightarrow 2$) is possible. We study the asymptotics of the survival probability in critical and subcritical cases.

1 Introduction

We study a two-type branching process in random environments. It can be viewed as a stochastic model for the sizes of a geographically structured population occupying two islands. Time is assumed discrete, so that one unit of time represents a generation of individuals, some living on island 1 and others on island 2. Those on island 1 give birth under influence of a randomly changing environment. They may migrate to island 2 immediately after birth, with a probability again depending upon the current environmental state. Individuals on island 2 do not migrate and their reproduction law is not influenced by any changing environment. Our main concern is the survival probability of the whole population.

An alternative interpretation of the model under study might be a population (type 1) subject to a changing environment, say in the form of a predator population of stationary but variable size. Its individuals may mutate into a second type, no longer exposed to the environmental variation (the predators, e. g.).

The model framework is that furnished by Bienaymé-Galton-Watson (BGW) processes with individuals living one unit of time and replaced by random numbers of offspring which are conditionally independent given the current state of the environment. We refer to such individuals as particles in order to emphasize

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the simplicity of their lives. Particles of type 1 and 2 are distinguished according to the island number they are occupying at the moment of observation. Our main assumptions are:

- particles of type 1 form a critical or subcritical branching process in a random environment,
- particles of type 2 form a critical branching process which is independent of the environment.

In Section 2 we recall known facts for constant environments. They will then be compared to the results of this paper on random environments. In Section 3.1 we describe IID environments (Independent and Identically Distributed environmental states), and then in Section 3.2 Markovian environments.

Our first result on the asymptotics of the survival probability, Theorem 1 is stated and proved in Section 4. It deals with the critical case with an IID environment. Section 5 contains our result for the subcritical case and an IID environment. The last two Sections, 6 and 7, expand our results for the IID case to the Markovian setting.

Use of notation. In asymptotic formulae constants denoted by the same letter c are always assumed to be fixed, that is independent of the parameter that tends to infinity (or zero).

2 Two-type decomposable branching processes

Consider a two-type BGW-process initiated at time zero by a single individual of type 1. In this paper we focus on the decomposable case where type 1 particles may produce particles of types 1 and 2 while the type 2 particles can give birth only to type 2 particles. Put

- X_n = the number of particles of type 1 present at time n , in particular, $X_0 = 1$,
- Y_n = the number of type 2 daughters produced by X_n particles of type 1,
- T = the first time n when $X_n = 0$,
- Z_n = the number of particles of type 2 present at time n , as a rule, we assume that $Z_0 = 0$ and, as a result, $Z_1 = Y_0$,
- $S_n = \sum_{k=0}^{n-1} X_k$ so that S_T gives the total number ever of particles of type 1.
- $W_n = \sum_{k=0}^{n-1} Y_k$ so that W_T gives the total number of type 2 daughters produced by all S_T particles of type 1.

The aim of this section is to summarize what is already known about such branching processes in the case of a constant environment. This will pave our way in terms of notation and basic manipulation with generating functions towards branching processes in IID random and then Markovian environments.

If the environment is constant from generation to generation, two-type decomposable BGW-processes are fully described by a pair of probability generating functions

$$\begin{aligned} f(s_1, s_2) &= \mathbf{E} \left[s_1^{\xi_1} s_2^{\xi_2} \right], \\ h(s) &= \mathbf{E} [s^\eta], \end{aligned}$$

where ξ_1 and ξ_2 represent the numbers of daughters of type 1 and 2 of a mother of type 1, while η stands for the number of daughters (necessarily of type 2) of a mother of type 2. Let

$$\begin{aligned} \mu_1 &= \mathbf{E} [\xi_1] = \frac{\partial f(s_1, s_2)}{\partial s_1} \Big|_{s_1=s_2=1}, \\ \mu_2 &= \mathbf{E} [\xi_1(\xi_1 - 1)] = \frac{\partial^2 f(s_1, s_2)}{\partial s_1^2} \Big|_{s_1=s_2=1}, \\ \theta_1 &= \mathbf{E} [\xi_2] = \frac{\partial f(s_1, s_2)}{\partial s_2} \Big|_{s_1=s_2=1}, \\ \theta_2 &= \mathbf{E} [\xi_2(\xi_2 - 1)] = \frac{\partial^2 f(s_1, s_2)}{\partial s_2^2} \Big|_{s_1=s_2=1}, \\ m_1 &= \mathbf{E} [\eta] = h'(1), \\ m_2 &= \mathbf{E} [\eta(\eta - 1)] = h''(1), \end{aligned}$$

be the first two moments of the reproduction laws. Concerning the second type of particles we will assume that

$$m_1 = 1, \quad m_2 \in (0, \infty), \tag{1}$$

implying that the probability of extinction

$$Q_n = \mathbf{P}(Z_n = 0 | X_0 = 0, Z_0 = 1)$$

(of a single-type BGW-process evolving in constant environment with the probability generating function $h(s)$) satisfies

$$1 - Q_n \sim \frac{2}{m_2 n}, \quad n \rightarrow \infty. \tag{2}$$

It follows that

$$a_n := -\log f(1, Q_n) \sim 1 - f(1, Q_n) \sim \frac{2\theta_1}{m_2 n}, \quad n \rightarrow \infty. \tag{3}$$

We will be interested in two kinds of reproduction regimes for particles of type 1, critical and subcritical. In the constant environment setting with $\mu_2 \in (0, \infty)$,

the critical case corresponds to $\mu_1 = 1$ and the subcritical case is given by $\mu_1 \in (0, 1)$. In the critical case with a constant environment we have

$$\mathbf{P}(X_n > 0) = \mathbf{P}(T > n) \sim \frac{2}{\mu_2 n}, \quad n \rightarrow \infty, \quad (4)$$

and according to Theorem 1 of [12]

$$\mathbf{P}(X_n + Z_n > 0) \sim \mathbf{P}(Z_n > 0) \sim \frac{2\sqrt{\theta_1}}{\sqrt{m_2\mu_2 n}}, \quad n \rightarrow \infty. \quad (5)$$

Next we outline a proof of (5) based on the representation

$$\begin{aligned} \mathbf{P}(Z_n > 0) &= \mathbf{E} \left[1 - \prod_{k=0}^{n-1} Q_{n-k}^{Y_k} \right] = \mathbf{E} \left[1 - \prod_{k=0}^{n-1} f^{X_k}(1, Q_{n-k}) \right] \\ &= \mathbf{E} \left[1 - e^{-\sum_{k=0}^{n-1} X_k a_{n-k}} \right], \end{aligned} \quad (6)$$

preparing for the proof in the random environment case, to be given in Section 4. Thanks to (4) and

$$\mathbf{P}(Z_n > 0) \leq \mathbf{P}(X_n + Z_n > 0) \leq \mathbf{P}(X_n > 0) + \mathbf{P}(Z_n > 0) \quad (7)$$

in order to prove (5) it is enough to verify that

$$\mathbf{P}(Z_n > 0) \sim \frac{2\sqrt{\theta_1}}{\sqrt{m_2\mu_2 n}}, \quad n \rightarrow \infty.$$

However, by the branching property the total progeny of a single-type branching process S_T is 1 plus ξ_1 independent daughter copies of S_T . In terms of the Laplace transform

$$\phi(\lambda) = e^{-\lambda} f(\phi(\lambda), 1),$$

where $\phi(\lambda) = \mathbf{E}e^{-\lambda S_T}$. As $\lambda \rightarrow 0$ Taylor expansion yields

$$1 - \phi(\lambda) = 1 - e^{-\lambda} + e^{-\lambda} \mu_1 (1 - \phi(\lambda)) - e^{-\lambda} \frac{\mu_2}{2} (1 - \phi(\lambda))^2 (1 + o(1)). \quad (8)$$

For $\mu_1 = 1$, this implies - after some calculation - that

$$1 - \phi(\lambda) \sim \sqrt{2\lambda/\mu_2}, \quad \lambda \rightarrow 0.$$

Thus, due to (3)

$$\mathbf{E} \left[1 - e^{-S_T a_n} \right] \sim \frac{2\sqrt{\theta_1}}{\sqrt{m_2\mu_2 n}}, \quad n \rightarrow \infty,$$

and it remains to verify, see (6), that

$$\mathbf{E} \left[1 - e^{-\sum_{k=0}^{n-1} X_k a_{n-k}} \right] \sim \mathbf{E} \left[1 - e^{-S_T a_n} \right], \quad n \rightarrow \infty.$$

This holds, indeed, since by (4) and for any fixed $\epsilon > 0$ the probability $\mathbf{P}(T > n\epsilon)$ is much smaller than the target value of order c/\sqrt{n} . (In [19] and [20] infinite second moments in decomposable two-type critical processes were allowed.)

On the other hand, in the subcritical case (8) implies that

$$1 - \phi(\lambda) \sim \lambda/(1 - \mu_1), \quad \lambda \rightarrow 0,$$

so that by (3)

$$\mathbf{E} [1 - e^{-S_T a_n}] \sim \frac{2\theta_1}{m_2(1 - \mu_1)n}, \quad n \rightarrow \infty.$$

In view of $\mathbf{P}(X_n > 0) \sim c\mu_1^n$ we conclude that in the subcritical case

$$\mathbf{P}(X_n + Z_n > 0) \sim \mathbf{P}(Z_n > 0) \sim \frac{2\theta_1}{m_2(1 - \mu_1)n}, \quad n \rightarrow \infty. \quad (9)$$

See [15] for a comprehensive study of subcritical decomposable branching processes in a constant environment.

3 Branching processes in a random environment

A randomly changing environment for BGW-processes is modeled by a random sequence of probability generating functions for the offspring distributions of consecutive generations. Throughout this paper we assume that the offspring distribution for type 2 particles is the same across the different states of the environment and characterized by the same generating function $h(s)$. This restriction greatly simplifies analysis still allowing new interesting asymptotic regimes.

We consider two types of stationarily changing environments: IID and Markovian.

3.1 IID environment

We start our description of the IID environment case by a simple illustration based on just two alternative bivariate generating functions $f^{(1)}(s_1, s_2)$ and $f^{(2)}(s_1, s_2)$ with mean offspring numbers $(\mu_1^{(1)}, \theta_1^{(1)})$ and $(\mu_1^{(2)}, \theta_1^{(2)})$ respectively. We assume that at each time n the environment is say "good" with probability π_1 , so that the type 1 particles reproduce independently according to $f^{(1)}(s_1, s_2)$, and with probability $\pi_2 = 1 - \pi_1$ the environment is "bad" and particles of type 1 reproduce according to the $f^{(2)}(s_1, s_2)$ law. In other words, the generating function $f(s_1, s_2)$ should be treated as a random function taking the form $f^{(1)}(s_1, s_2)$ or $f^{(2)}(s_1, s_2)$ with probabilities π_1 and π_2 . In particular, the vector of the mean offspring numbers (μ_1, θ_1) takes values $(\mu_1^{(1)}, \theta_1^{(1)})$ and $(\mu_1^{(2)}, \theta_1^{(2)})$ with probabilities π_1 and π_2 .

More generally, our two-type branching process in an IID random environment is characterized (besides the fixed reproduction law $h(s)$ for the type 2

particles) by a sequence of generating functions $\{f_n(s_1, s_2)\}_{n=0}^\infty$ independently drawn from a certain distribution over probability generating functions so that

$$f_n(s_1, s_2) \stackrel{d}{=} f(s_1, s_2). \quad (10)$$

In this setting the respective conditional moments μ_1 , μ_2 , θ_1 , and θ_2 should be treated as random variables. An important role is played by the random variable $\zeta = \log \mu_1$ representing the step size of the so-called associated random walk [3] formed by the partial sums $\zeta_0 + \dots + \zeta_{n-1}$ with $\zeta_i \stackrel{d}{=} \zeta$.

Use of notation. Characteristics of the reproduction law in generation k are denoted by adding an extra lower index k to the generic notation, like in (10). Another example: $(\theta_{1,k}, \theta_{2,k})$ in (28) represent the conditional moments (θ_1, θ_2) for the offspring numbers of the type 1 particles living at time k . (11)

3.2 Markovian environment

One way to relax the IID assumption on the environment is to allow for Markovian dependence among its consecutive states. We implement this idea by modeling changes in terms of an irreducible positively recurrent Markov chain $\{e_n\}_{n=0}^\infty$ with countably many states $\{1, 2, \dots\}$. Assuming a stationary initial distribution, we will associate with each state i of this chain a probability generating function $f^{(i)}(s_1, s_2)$, so that the changing environment for the branching process is governed by the sequence of identically reproduction laws

$$f_n(s_1, s_2) = f^{(e_n)}(s_1, s_2), \quad n = 0, 1, \dots$$

with Markovian dependence. Due to the stationarity we can again write (10) and use the same notation for the marginal moments of the reproduction laws as in the IID case.

To build a bridge to the IID environment case we use an embedding through a sequence of regeneration moments $\{\tau_k\}_{k=0}^\infty$ defined as

$$\tau_0 := 0, \tau_{k+1} := \min\{n > \tau_k : e_n = e_0\}. \quad (12)$$

The times $\tau_{k+1} - \tau_k$ between consecutive regenerations are independent and all distributed as $\tau := \tau_1$. The embedded process (X_n^*, Z_n^*) defined as

$$(X_n^*, Z_n^*) = (X_{\tau_n}, Z_{\tau_n}), \quad n = 0, 1, \dots$$

is a decomposable branching process in an IID environment with two types of particles 1^* and 2^* and conditional reproduction generating functions

$$f^*(s_1, s_2) := f^{(e_0)}(f^{(e_1)}(\dots(f^{(e_{\tau-1})}(s_1, s_2), h(s_2))\dots), h_{\tau-1}(s_2)), \quad (13)$$

$$h^*(s) := h(h(\dots h(s)\dots)) = h_\tau(s), \quad (14)$$

where $h_k(s)$ stands for the k -fold iteration of $h(s)$.

The key difference from the IID case is that the reproduction law for the 2*-type particles being dependent on the random environment. However, this dependence is of specific nature which we are able to manage using the law of large numbers for the renewal process. Notice that on its own the 2*-type particles form a so-called degenerate critical branching process in an IID random environment [3]: its conditional offspring mean is deterministic $m_1^* = 1$. Meanwhile, the conditional variance is random $m_2^* = \tau m_2$.

Taking the first and second order derivatives of (13), we can express the moments of the star-reproduction law in terms of the moments of the consecutive reproduction laws with Markovian dependence. In what follows we use (11) again, while keeping in mind that the sequence $(\mu_{1,k}, \mu_{2,k}, \theta_{1,k}, \theta_{2,k})_{k=0}^{\tau-1}$ now consists of dependent random vectors. It can be shown that

$$\begin{aligned}\mu_1^* &= \prod_{k=0}^{\tau-1} \mu_{1,k}, & \mu_2^* &= \mu_1^* \sum_{k=0}^{\tau-1} \frac{\mu_{2,k}}{\mu_{1,k}} \prod_{i=k+1}^{\tau-1} \mu_{1,i}, \\ \theta_1^* &= \sum_{k=0}^{\tau-1} \theta_{1,k} \prod_{i=0}^{k-1} \mu_{1,i},\end{aligned}$$

Furthermore, setting

$$A_{k,n} = \sum_{j=k}^n \theta_{1,j} \prod_{i=k}^{j-1} \mu_{1,i}$$

we can write

$$\begin{aligned}\theta_2^* &= \sum_{k=0}^{\tau-1} \theta_{2,k} \prod_{i=0}^{k-1} \mu_{1,i} \\ &+ \sum_{k=0}^{\tau-2} (\mu_{2,k} A_{k+1,\tau-1}^2 + 2\mu_{1,k} \theta_{1,k} A_{k+1,\tau-1} + \sigma^2 (\tau - 1 - k) \theta_{1,k}) \prod_{i=0}^{k-1} \mu_{1,i}.\end{aligned}$$

Considering

$$\zeta^* = \sum_{k=0}^{\tau-1} \zeta_k, \quad \zeta_k = \log \mu_{1,k} = \zeta(e_k),$$

observe that due to stationarity of the Markovian environment we can use a version of the Wald identity

$$\mathbf{E}[\zeta^*] = \mathbf{E}[\tau] \mathbf{E}[\zeta] \tag{15}$$

provided

$$\sum_{k=0}^{\infty} \mathbf{E}[|\zeta_k| 1_{\{\tau \geq k+1\}}] < \infty. \tag{16}$$

Equation (15) follows from

$$\begin{aligned}\mathbf{E}[\zeta^*] &= \sum_{k=1}^{\infty} \sum_{n=0}^{k-1} \mathbf{E}[\zeta_n 1_{\{\tau=k\}}] = \sum_{n=0}^{\infty} \mathbf{E}[\zeta(e_n) 1_{\{\tau>n\}}] \\ &= \sum_{n=0}^{\infty} \sum_{i=1}^{\infty} \zeta(i) \mathbf{P}(e_n = i, \tau > n) = \sum_{i=1}^{\infty} \zeta(i) \pi_i \mathbf{E}[\tau] = \mathbf{E}[\tau] \mathbf{E}[\zeta],\end{aligned}$$

where π_i are the stationary probabilities and the second last equality is justified as follows. According to Theorem 6.5.2 from [10] for any state j

$$\mu_j(i) = \sum_{n=0}^{\infty} \mathbf{P}(e_n = i, \tau > n | e_0 = j)$$

defines a stationary measure which is necessarily of the form $\mu_j(i) = c_j \pi_i$, where

$$\begin{aligned}\sum_{j=1}^{\infty} \pi_j c_j &= \sum_{j=1}^{\infty} \pi_j \sum_{i=1}^{\infty} \mu_j(i) = \sum_{j=1}^{\infty} \pi_j \sum_{n=0}^{\infty} \sum_{i=1}^{\infty} \mathbf{P}(e_n = i, \tau > n | e_0 = j) \\ &= \sum_{j=1}^{\infty} \pi_j \mathbf{E}[\tau | e_0 = j] = \mathbf{E}[\tau].\end{aligned}$$

Developing the example of two environmental states from Section 3.1 let us consider a Markov chain $\{e_n\}_{n=0}^{\infty}$ with transition probabilities

$$\begin{pmatrix} 1 - d\pi_2 & d\pi_2 \\ d\pi_1 & 1 - d\pi_1 \end{pmatrix}, \quad 0 < d < \min\left(\frac{1}{\pi_1}, \frac{1}{\pi_2}\right)$$

and a stationary distribution (π_1, π_2) . (Notice that $d = 1$ corresponds to the IID case.) Assuming stationarity we get for the regeneration time

$$\begin{aligned}\mathbf{P}(\tau = 1) &= \pi_1(1 - d\pi_2) + \pi_2(1 - d\pi_1) = 1 - 2\pi_1\pi_2d, \\ \mathbf{P}(\tau = k) &= \pi_1 d\pi_2(1 - d\pi_1)^{k-2} d\pi_1 + \pi_2 d\pi_1(1 - d\pi_2)^{k-2} d\pi_2 \\ &= d\pi_1\pi_2 \left(d\pi_1(1 - d\pi_1)^{k-2} + d\pi_2(1 - d\pi_2)^{k-2} \right), \quad k \geq 2,\end{aligned}$$

implying

$$\mathbf{E}[\tau - 1] = 1, \quad \mathbf{E}[\tau(\tau - 1)] = \frac{2}{d\pi_1\pi_2} - \frac{4}{d}.$$

If (b_1, b_2) are the two possible values for ζ , we can write $\mathbf{E}[\zeta] = \pi_1 b_1 + \pi_2 b_2$ and

$$\begin{aligned}\mathbf{E}[\zeta^*] &= \mathbf{E}[\mathbf{E}[\zeta^* | \tau]; \tau = 1] + \mathbf{E}[\mathbf{E}[\zeta^* | \tau]; \tau \geq 2] \\ &= \pi_1(1 - d\pi_2)b_1 + \pi_2(1 - d\pi_1)b_2 \\ &\quad + \pi_1 d\pi_2 \left(b_1 + \frac{b_2}{d\pi_1} \right) + \pi_2 d\pi_1 \left(b_2 + \frac{b_1}{d\pi_2} \right) = 2\mathbf{E}[\zeta],\end{aligned}$$

confirming equality (15).

4 Critical processes in IID environment

The single type critical branching process with an IID environment displays an asymptotic behavior that is in stark contrast with the constant environment formula (4). According to Theorem 1 in [13], if

$$\mathbf{E}[\zeta] = 0, \quad \text{Var}[\zeta] \in (0, \infty), \quad (17)$$

$$\mathbf{E}[\mu_2 \mu_1^{-2} (1 + \max(0, \log \mu_1))] < \infty, \quad (18)$$

then

$$\mathbf{P}(X_n > 0) = \mathbf{P}(T > n) \sim \frac{c}{\sqrt{n}}, \quad n \rightarrow \infty. \quad (19)$$

(A much more general limit theorem is obtained in [3].) The following theorem shows that in the decomposable case the difference between the constant and random environments is even more striking. For constant environments the survival probability decays as c/\sqrt{n} , see (5), but in random environments the decay is like $c/\log n$.

Theorem 1 *Consider a critical decomposable branching process in an IID environment satisfying (1), (17), (18), and*

$$\mathbf{E}[\mu_1^{-1}] < \infty. \quad (20)$$

If for some positive α

$$\mathbf{P}(\theta_1 < 1/x) = o((\log x)^{-3-\alpha}), \quad x \rightarrow \infty, \quad (21)$$

$$\mathbf{P}(\theta_1 > x) = o((\log x)^{-3-\alpha}), \quad x \rightarrow \infty, \quad (22)$$

$$\mathbf{P}(\theta_2 > x\theta_1) = o((\log x)^{-3-\alpha}), \quad x \rightarrow \infty, \quad (23)$$

then there exists a constant K_0 such that

$$\mathbf{P}(X_n + Z_n > 0) \sim \mathbf{P}(Z_n > 0) \sim \frac{K_0}{\log n}, \quad n \rightarrow \infty. \quad (24)$$

Before the proof, we make some comments on the conditions and statement of this theorem.

Use of notation. From now on we will often use abbreviations $x_a = (\log x)^{2+a}$ and $n_a = (\log n)^{2+a}$. (25)

Conditions (21), (22), and (23) are needed for the following properties to hold for any fixed $\varepsilon > 0$

$$\mathbf{P}\left(\min_{0 \leq k \leq x_\alpha} \theta_{1,k} < x^{-\varepsilon}\right) = o\left(\frac{1}{\log x}\right), \quad x \rightarrow \infty, \quad (26)$$

$$\mathbf{P}\left(\max_{0 \leq k \leq x_\alpha} \theta_{1,k} > x^\varepsilon\right) = o\left(\frac{1}{\log x}\right), \quad x \rightarrow \infty, \quad (27)$$

$$\mathbf{P}\left(\max_{0 \leq k \leq x_\alpha} (\theta_{2,k}/\theta_{1,k}) > x^\varepsilon\right) = o\left(\frac{1}{\log x}\right), \quad x \rightarrow \infty. \quad (28)$$

Each of them is proven via intermediate step like

$$\mathbf{P} \left(\min_{0 \leq k \leq x_\alpha} \theta_{1,k} < x^{-\varepsilon} \right) \leq x_\alpha \mathbf{P} (\theta_1 < x^{-\varepsilon})$$

relying on the IID assumption for consecutive environmental states. The constant K_0 in the statement (24) is the same as in the asymptotic formula from [1]

$$\mathbf{P} (S_T > x) \sim \frac{K_0}{\log x}, \quad x \rightarrow \infty. \quad (29)$$

This constant has a complicated nature and is not further explained here. It is necessary to mention that the representation (29) has been proved in [1] under conditions (17), (18), and (20) only for the case when the probability generating functions $f_n(s, 1)$ are linear-fractional with probability 1. However, the latter restriction is easily removed using the results established later on for the general case in [13] and [3].

We now turn to the proof of Theorem 1. To derive (24) it suffices, in view of (19), to show that

$$\mathbf{P} (Z_n > 0) \sim \frac{K_0}{\log n}, \quad n \rightarrow \infty,$$

or

$$\liminf_{n \rightarrow \infty} \{\log n \cdot \mathbf{P}(Z_n > 0)\} \geq K_0, \quad (30)$$

$$\limsup_{n \rightarrow \infty} \{\log n \cdot \mathbf{P}(Z_n > 0)\} \leq K_0. \quad (31)$$

These inequalities will be established next using a counterpart of (6)

$$\begin{aligned} \mathbf{P} (Z_n > 0) &= \mathbf{E} \left[1 - \prod_{k=0}^{n-1} Q_{n-k}^{Y_k} \right] = \mathbf{E} \left[1 - \prod_{k=0}^{n-1} f_k^{X_k}(1, Q_{n-k}) \right] \\ &= \mathbf{E} \left[1 - \exp \left\{ \sum_{k=0}^{n-1} X_k \log f_k(1, Q_{n-k}) \right\} \right], \end{aligned} \quad (32)$$

and the following lemma.

Lemma 2 *Consider conditional moments of the entity W_n defined at the beginning of Section 2:*

$$S_n^{(i)} = \sum_{k=0}^{n-1} X_k \theta_{i,k}, \quad i = 1, 2.$$

Under conditions (17), (21), (22), and (23) we have

$$\mathbf{P} (S_T^{(1)} > x) \sim \frac{K_0}{\log x}, \quad x \rightarrow \infty,$$

and for any fixed $\epsilon > 0$, see (25),

$$\mathbf{P}\left(S_T^{(2)} > n^\epsilon S_T^{(1)}; T \leq n_\alpha\right) = o\left(\frac{1}{\log n}\right), \quad n \rightarrow \infty.$$

PROOF OF LEMMA 2 . For any fixed $\epsilon > 0$ we have

$$\begin{aligned} \mathbf{P}\left(S_T^{(1)} > x\right) &\geq \mathbf{P}\left(S_T^{(1)} > x; T \leq x_\alpha; \min_{0 \leq k \leq T} \theta_{1,k} > x^{-\epsilon}\right) \\ &\geq \mathbf{P}\left(S_T > x^{1+\epsilon}\right) - \mathbf{P}\left(T > x_\alpha\right) - \mathbf{P}\left(\min_{0 \leq k \leq x_\alpha} \theta_{1,k} \leq x^{-\epsilon}\right). \end{aligned}$$

Notice that according to (19)

$$\mathbf{P}\left(T > (\log x)^{2+\epsilon}\right) = o\left(\frac{1}{\log x}\right), \quad x \rightarrow \infty, \quad \text{for any fixed } \epsilon > 0. \quad (33)$$

Thus, using (26) and (29) we get

$$\liminf_{x \rightarrow \infty} \left\{ \log x \cdot \mathbf{P}\left(S_T^{(1)} > x\right) \right\} \geq \liminf_{x \rightarrow \infty} \left\{ \log x \cdot \mathbf{P}\left(S_T > x^{1+\epsilon}\right) \right\} \geq K_0/(1+\epsilon).$$

To obtain a similar estimate from above we write, recall (25),

$$\begin{aligned} \mathbf{P}\left(S_T^{(1)} > x\right) &\leq \mathbf{P}\left(S_T^{(1)} > x; T \leq x_\alpha; \max_{0 \leq k \leq T} \theta_{1,k} \leq x^\epsilon\right) \\ &\quad + \mathbf{P}\left(T > x_\alpha\right) + \mathbf{P}\left(T \leq x_\alpha; \max_{0 \leq k \leq T} \theta_{1,k} > x^\epsilon\right) \\ &\leq \mathbf{P}\left(S_T > x^{1-\epsilon}\right) + \mathbf{P}\left(T > x_\alpha\right) + \mathbf{P}\left(\max_{0 \leq k \leq x_\alpha} \theta_{1,k} > x^\epsilon\right), \end{aligned}$$

which together with (27), (29), and (33) yields

$$\begin{aligned} \limsup_{x \rightarrow \infty} \left\{ \log x \cdot \mathbf{P}\left(S_T^{(1)} > x\right) \right\} &\leq \limsup_{x \rightarrow \infty} \left\{ \log x \cdot \mathbf{P}\left(S_T > x^{1-\epsilon}\right) \right\} \\ &\leq K_0/(1-\epsilon). \end{aligned}$$

Finally, according to (28)

$$\begin{aligned} \mathbf{P}\left(S_T^{(2)} > n^\epsilon S_T^{(1)}; T \leq n_\alpha\right) &\leq \mathbf{P}\left(\max_{1 \leq k \leq T} (\theta_{2,k}/\theta_{1,k}) > n^\epsilon; T \leq n_\alpha\right) \\ &= o\left(\frac{1}{\log n}\right), \quad n \rightarrow \infty. \end{aligned}$$

□

PROOF of (30). It follows from (32) and monotonicity of Q_n that for any fixed $\epsilon \in (0, 1)$

$$\mathbf{P}\left(Z_n > 0\right) \geq \mathbf{E}\left[1 - \exp\left\{\sum_{k=0}^{T-1} X_k \log f_k(1, Q_n)\right\}; S_T^{(2)} \leq n^\epsilon S_T^{(1)}, T \leq n_\alpha\right].$$

Observe that $\log(1-x) \leq -x$ and

$$f(1, s) \leq 1 + \theta_1 (s-1) + \frac{\theta_2}{2} (1-s)^2,$$

which holds due to monotonicity of the second derivative of the generating function. Therefore,

$$\log f(1, s) \leq -\theta_1 (1-s) + (\theta_2/2) (1-s)^2$$

and it follows from (2) that given $S_T^{(2)} \leq n^\varepsilon S_T^{(1)}$

$$\sum_{k=0}^{T-1} X_k \log f_k(1, Q_n) \leq -cn^{-1} S_T^{(1)}$$

for sufficiently large n . As a result we see that for large n

$$\begin{aligned} \mathbf{P}(Z_n > 0) &\geq \mathbf{E} \left[1 - e^{-cn^{-1} S_T^{(1)}}; S_T^{(2)} \leq n^\varepsilon S_T^{(1)}, T \leq n_\alpha \right] \\ &\geq \mathbf{E} \left[1 - e^{-cn^{-1} S_T^{(1)}} \right] - \mathbf{P}(T > n_\alpha) - \mathbf{P} \left(S_T^{(2)} > n^\varepsilon S_T^{(1)}; T \leq n_\alpha \right). \end{aligned}$$

In view of (33) and Lemma 2 it remains to observe that for any positive λ

$$\mathbf{E} \left[1 - e^{-\lambda S_T^{(1)}} \right] \sim \frac{K_0}{\log(1/\lambda)}, \quad \lambda \rightarrow 0,$$

which again due to Lemma 2 follows from the Tauberian Theorem 4 in [11, Ch. XIII.5] applied to the right hand side of

$$\lambda^{-1} \mathbf{E} \left[1 - e^{-\lambda S_T^{(1)}} \right] = \int_0^\infty \mathbf{P} \left(S_T^{(1)} > x \right) e^{-\lambda x} dx.$$

□

PROOF of (31). In a similar manner the inequality

$$\mathbf{P}(Z_n > 0) \leq \mathbf{E} \left[1 - \exp \left\{ \sum_{k=0}^{T-1} X_k \log f_k(1, Q_{n-T}) \right\} \right]$$

entails that for all sufficiently large n

$$\begin{aligned} &\mathbf{E} \left[1 - \exp \left\{ \sum_{k=0}^{T-1} X_k \log f_k(1, Q_{n-T}) \right\}; T \leq n_\alpha; \max_{0 \leq k \leq T} \theta_{1,k} \leq n^\varepsilon \right] \\ &\leq \mathbf{E} \left[1 - \exp \left\{ \sum_{k=0}^{T-1} X_k \log(1 - c\theta_{1,k} n^{-1}) \right\}; T \leq n_\alpha; \max_{0 \leq k \leq T} \theta_{1,k} \leq n^\varepsilon \right] \\ &\leq \mathbf{E} \left[1 - \exp \left\{ \sum_{k=0}^{T-1} X_k \log(1 - cn^{\varepsilon-1}) \right\} \right] \\ &\leq \mathbf{E} \left[1 - e^{-cn^{\varepsilon-1} S_T^{(1)}} \right]. \end{aligned}$$

Thus

$$\mathbf{P}(Z_n > 0) \leq \mathbf{E} \left[1 - e^{-cn^{\varepsilon-1} S_T^{(1)}} \right] + \mathbf{P}(T > n_\alpha) + \mathbf{P} \left(\max_{0 \leq k \leq n_\alpha} \theta_{1,k} > n^\varepsilon \right)$$

and (31) follows due to (27) and (33). \square

5 The subcritical case with an IID environment

We continue studying BGW-processes in IID environment but now we assume

$$\mathbf{E}[\zeta] < 0, \quad \text{Var}[\zeta] \in (0, \infty) \quad (34)$$

instead of (17). Results rely upon a theorem from [16] giving the asymptotics for $\mathbf{P}(W_T > x)$ as $x \rightarrow \infty$. It requires the important technical assumption of existence of a constant κ such that

$$\mathbf{E}[e^{\kappa\zeta}] = \mathbf{E}[\mu_1^\kappa] = 1, \quad 0 < \kappa < \infty. \quad (35)$$

If, in addition, for some $\delta > 0$

$$0 < \mathbf{E}[\xi_2^{\kappa+\delta}] < \infty, \quad \mathbf{E}[\theta_1^\kappa] < \infty, \quad (36)$$

and

$$\{\kappa > 1, \mathbf{E}[|\xi_1|^\kappa] < \infty\} \cup \{\kappa \leq 1, \mathbf{E}[|\mu_2 - \mu_1^2|^\kappa + |\theta_2 - \theta_1^2|^\kappa] < \infty\}, \quad (37)$$

then, according to [16], there exists a constant $C_\kappa \in (0, \infty)$ such that

$$\mathbf{P}(W_T > x) \sim C_\kappa x^{-\kappa}, \quad x \rightarrow \infty. \quad (38)$$

Theorem 3 *If conditions (34), (35), (36), (37) hold, then*

$$\mathbf{P}(X_n + Z_n > 0) \sim \mathbf{P}(Z_n > 0) \sim K_\kappa \cdot q_\kappa(n), \quad n \rightarrow \infty, \quad (39)$$

for some positive constant K_κ , given by (41) below, where

$$q_\kappa(n) = \begin{cases} n^{-\kappa}, & \text{if } \kappa < 1, \\ n^{-1} \log n, & \text{if } \kappa = 1, \\ n^{-1}, & \text{if } \kappa > 1. \end{cases} \quad (40)$$

PROOF Starting from the first equality in (32) we obtain for sufficiently large n

$$\begin{aligned} \mathbf{P}(Z_n > 0) &\geq \mathbf{E} \left[1 - \prod_{k=0}^{n-1} Q_n^{Y_k}; T \leq N \log n \right] \\ &= \mathbf{E} [1 - e^{W_T \log Q_n}; T \leq N \log n] \\ &\geq \mathbf{E} [1 - e^{W_T \log Q_n}] - \mathbf{P}(T > N \log n), \end{aligned}$$

where a positive constant N will be specified later. On the other hand, we have a similar upper bound

$$\begin{aligned}\mathbf{P}(Z_n > 0) &\leq \mathbf{E}\left[1 - Q_{n-T}^{W_T}; T \leq N \log n\right] + \mathbf{P}(T > N \log n) \\ &\leq \mathbf{E}\left[1 - e^{W_T \log Q_{n-N \log n}}\right] + \mathbf{P}(T > N \log n).\end{aligned}$$

Now observe that due to (2) and (38), the same Tauberian theorem used in the critical case yields

$$\mathbf{E}\left[1 - e^{W_T \log Q_n}\right] \sim K_\kappa \cdot q_\kappa(n), \quad n \rightarrow \infty$$

with

$$K_\kappa = \begin{cases} \Gamma(1 - \kappa) C_\kappa \left(\frac{2}{m_2}\right)^\kappa, & \text{if } \kappa < 1, \\ \frac{2}{m_2} C_1, & \text{if } \kappa = 1, \\ \frac{2}{m_2} \int_0^\infty \mathbf{P}(W_T > x) dx, & \text{if } \kappa > 1. \end{cases} \quad (41)$$

It remains to use the fact that (see [4], [5], [6]) under (34) and (35)

$$\mathbf{P}(T > n) = o(A^n) \text{ for some constant } A \in (0, 1), \quad (42)$$

if we choose $N = 2/\log(A^{-1})$. □

6 The critical case with a Markovian environment

As compared to the IID case. Markovian environments require extra conditions on the underlying Markov chain. In particular, we assume (16) and that for some $\rho > 0$

$$\mathbf{P}(\tau > x) = o\left(\frac{1}{x \log^{1+\rho} x}\right), \quad x \rightarrow \infty. \quad (43)$$

This implies that $a := \mathbf{E}[\tau] < \infty$ and due to (15) conditions $\mathbf{E}[\zeta^*] = 0$ and $\mathbf{E}[\zeta] = 0$ become equivalent.

Moreover, under condition (43) the sequence of regeneration times (12) satisfies

$$\mathbf{P}(|k^{-1}\tau_k - a| > \varepsilon) = o((\log k)^{-1-\rho}), \quad k \rightarrow \infty, \quad (44)$$

for an arbitrarily small $\varepsilon > 0$, cf. [14].

Theorem 4 *Assume (16), (43) and*

$$\mathbf{E}[\zeta] = 0, \quad \text{Var}[\zeta^*] \in (0, \infty), \quad (45)$$

$$\mathbf{E}\left[\mu_2^* (\mu_1^*)^{-2} (1 + \max(0, \log \mu_1^*))\right] < \infty, \quad \mathbf{E}\left[(\mu_1^*)^{-1}\right] < \infty.$$

Let, further, for some positive α

$$\mathbf{P}(\theta_1^* < 1/x) = o((\log x)^{-3-\alpha}), \quad x \rightarrow \infty, \quad (46)$$

$$\mathbf{P}(\theta_1^* > x) = o((\log x)^{-3-\alpha}), \quad x \rightarrow \infty, \quad (47)$$

$$\mathbf{P}(\theta_2^* > x\theta_1^*) = o((\log x)^{-3-\alpha}), \quad x \rightarrow \infty. \quad (48)$$

Then there exists a constant $K^* > 0$ such that

$$\mathbf{P}(X_n + Z_n > 0) \sim \mathbf{P}(Z_n > 0) \sim \frac{K^*}{\log n}, \quad n \rightarrow \infty. \quad (49)$$

PROOF The statement is derived in two steps: first

$$\mathbf{P}(X_r^* + Z_r^* > 0) \sim \mathbf{P}(Z_r^* > 0) \sim \frac{K^*}{\log r}, \quad r \rightarrow \infty \quad (50)$$

and then (49).

Fix $\delta \in (0, 1/4)$ and write

$$\begin{aligned} \mathbf{P}(Z_r^* > 0) &= \mathbf{P}(Z_r^* > 0; \tau_{r^\delta} \leq r^{2\delta}, |\tau_r - ra| \leq r^{1-\delta}) \\ &\quad + O(\mathbf{P}(\tau_{r^\delta} > r^{2\delta})) + O(\mathbf{P}(|\tau_r - ra| > r^\delta)). \end{aligned}$$

Here the last two terms are treated with the help of $\mathbf{P}(\tau_{r^\delta} > r^{2\delta}) \leq ar^{-\delta}$ and (44), while the main term is analyzed by means of ideas from the proof of Theorem 1. Letting Y_k^* be the number of type 2* daughters produced by X_k^* particles of type 1* and putting $T^* = \min\{r : Z_r^* = 0\}$, we deduce from

$$\begin{aligned} &\mathbf{P}(Z_r^* > 0; \tau_{r^\delta} \leq r^{2\delta}, |\tau_r - ra| \leq r^{1-\delta}) \\ &= \mathbf{E} \left[1 - \prod_{k=0}^{r-1} Q_{\tau_r - \tau_k}^{Y_k^*}; \tau_{r^\delta} \leq r^{2\delta}, |\tau_r - ra| \leq r^{1-\delta} \right], \end{aligned}$$

a lower bound

$$\begin{aligned} &\mathbf{P}(Z_r^* > 0; \tau_{r^\delta} \leq r^{2\delta}, |\tau_r - ra| \leq r^{1-\delta}) \\ &\geq \mathbf{E} \left[1 - \prod_{k=0}^{T^*-1} Q_{\tau_r - \tau_k}^{Y_k^*}; T^* \leq r^\delta, \tau_{r^\delta} \leq r^{2\delta}, |\tau_r - ra| \leq r^{1-\delta} \right] \\ &\geq \mathbf{E} \left[1 - \prod_{k=0}^{T^*-1} f_k^{X_k^*}(1, Q_{2ra}); T^* \leq r^\delta, \tau_{r^\delta} \leq r^{2\delta}, |\tau_r - ra| \leq r^{1-\delta} \right] \\ &= \mathbf{E} \left[1 - \prod_{k=0}^{T^*-1} f_k^{X_k^*}(1, Q_{2ra}) \right] + O(\mathbf{P}(T^* > r^\delta)) + o\left(\frac{1}{\log^{1+\rho} r}\right). \end{aligned}$$

Hence, applying arguments used to demonstrate (30) in Lemma 1 one can show that

$$\liminf_{r \rightarrow \infty} \{\log r \cdot \mathbf{P}(Z_r^* > 0)\} \geq K^*.$$

By similar arguments one can show that

$$\limsup_{r \rightarrow \infty} \{\log r \cdot \mathbf{P}(Z_r^* > 0)\} \leq K^*,$$

proving (50) in view of

$$\mathbf{P}(X_r^* > 0) = O\left(\frac{1}{\sqrt{r}}\right), r \rightarrow \infty.$$

To demonstrate that (49) follows from (50) we set

$$N(n) := \max \{k : \tau_k \leq n\}.$$

Clearly,

$$\mathbf{P}\left(X_{N(n)+1}^* + Z_{N(n)+1}^* > 0\right) \leq \mathbf{P}(X_n + Z_n > 0) \leq \mathbf{P}\left(X_{N(n)}^* + Z_{N(n)}^* > 0\right), \quad (51)$$

and for any $\varepsilon \in (0, 1)$

$$\begin{aligned} \mathbf{P}\left(X_{N(n)}^* + Z_{N(n)}^* > 0\right) &= \mathbf{P}\left(X_{N(n)}^* + Z_{N(n)}^* > 0; N(n) \geq a^{-1}n(1-\varepsilon)\right) \\ &\quad + O\left(\mathbf{P}\left(N(n) < a^{-1}n(1-\varepsilon)\right)\right). \end{aligned}$$

For the first we have

$$\mathbf{P}\left(X_{N(n)}^* + Z_{N(n)}^* > 0; N(n) \geq a^{-1}n(1-\varepsilon)\right) \leq \mathbf{P}\left(X_{a^{-1}n(1-\varepsilon)}^* + Z_{a^{-1}n(1-\varepsilon)}^* > 0\right).$$

On the other hand, again by (44) as $n \rightarrow \infty$

$$\mathbf{P}\left(N(n) < a^{-1}n(1-\varepsilon)\right) = \mathbf{P}\left(S_{a^{-1}n(1-\varepsilon)} > n\right) = o\left(\frac{1}{\log^{1+\rho} n}\right).$$

Thus,

$$\begin{aligned} \limsup_{n \rightarrow \infty} \{\log n \cdot \mathbf{P}(X_n + Z_n > 0)\} \\ \leq \limsup_{n \rightarrow \infty} \left\{ \log n \cdot \mathbf{P}\left(X_{a^{-1}n(1-\varepsilon)}^* + Z_{a^{-1}n(1-\varepsilon)}^* > 0\right) \right\} \leq K^*. \end{aligned}$$

A similar estimate from below follows from

$$\begin{aligned} \mathbf{P}\left(X_{N(n)+1}^* + Z_{N(n)+1}^* > 0\right) \\ \geq \mathbf{P}\left(X_{a^{-1}n(1+\varepsilon)}^* + Z_{a^{-1}n(1+\varepsilon)}^* > 0; N(n) + 1 \leq a^{-1}n(1+\varepsilon)\right) \\ = \mathbf{P}\left(X_{a^{-1}n(1+\varepsilon)}^* + Z_{a^{-1}n(1+\varepsilon)}^* > 0\right) + o\left(\frac{1}{\log^{1+\rho} n}\right). \end{aligned}$$

□

7 Subcritical processes with a Markovian environment

Assume now that the two-type subcritical process (X_n, Z_n) evolves in a stationary Markovian random environment as defined in Section 3.2. Here, similarly to Section 6 the auxiliary branching process (X_r^*, Z_r^*) in IID environment with probability generating functions (13) and (14) plays an important role.

Theorem 5 *Assume that*

$$\mathbf{E}[\zeta] < 0, \quad \text{Var}[\zeta^*] \in (0, \infty), \quad (52)$$

and conditions (35), (36), (37) are valid for the corresponding random variables related to the embedded process (X_r^, Z_r^*) with the key constant κ replaced by κ^* . Suppose, in addition, that*

$$\mathbf{P}(\tau > x) = o\left(x^{-1-\min(\kappa^*, 1)}\right), \quad x \rightarrow \infty. \quad (53)$$

Then there exists a constant $K^ \equiv K^*(\kappa^*) > 0$ such that, as $n \rightarrow \infty$, see (40),*

$$\mathbf{P}(X_n + Z_n > 0) \sim K^* q_{\kappa^*}(n). \quad (54)$$

PROOF Our main arguments here are similar to that used in the proof of Theorem 4. Fix $\varepsilon \in (0, 1)$ and a sufficiently large N and with $a = \mathbf{E}[\tau]$ write

$$\begin{aligned} \mathbf{P}(Z_r^* > 0) &= \mathbf{P}(Z_r^* > 0; \mathcal{B}(r, \varepsilon)) \\ &\quad + O\left(\mathbf{P}\left(\tau_{N \log r} > r^{\kappa^*} \log^3 r\right) + \mathbf{P}(|\tau_r - ra| > r\varepsilon)\right). \end{aligned}$$

where

$$\mathcal{B}(r, \varepsilon) := \left\{ \tau_{N \log r} \leq r^{\kappa^*} \log^3 r, |\tau_r - ra| \leq \varepsilon r \right\}.$$

Clearly,

$$\mathbf{P}\left(\tau_{N \log r} > r^{\kappa^*} \log^3 r\right) \leq \frac{Na}{r^{\kappa^*} \log^2 r} = o\left(r^{-\kappa^*}\right). \quad (55)$$

Further, if $\kappa^* < 1$ then, according to [14] we have under condition (53),

$$\mathbf{P}(|\tau_r - ra| > \varepsilon r) = o\left(r^{-\kappa^*}\right). \quad (56)$$

Thus,

$$\begin{aligned}
\mathbf{P}(Z_r^* > 0) &\geq \mathbf{P}(Z_r^* > 0; \mathcal{B}(r, \varepsilon)) \\
&= \mathbf{E} \left[1 - \prod_{k=0}^{r-1} Q_{\tau_r - \tau_k}^{Y_k^*}; \mathcal{B}(r, \varepsilon) \right] \\
&\geq \mathbf{E} \left[1 - \prod_{k=0}^{T^*-1} Q_{\tau_r - \tau_k}^{Y_k^*}; T^* \leq N \log r; \mathcal{B}(r, \varepsilon) \right] \\
&\geq \mathbf{E} \left[1 - \prod_{k=0}^{T^*-1} Q_{ra+2\varepsilon r}^{Y_k^*}; T^* \leq N \log r; \mathcal{B}(r, \varepsilon) \right] \\
&= \mathbf{E} \left[1 - e^{W_T^* \log Q_{ra+2\varepsilon r}} \right] - \mathbf{P}(T^* > N \log r) - \mathbf{P}(\bar{\mathcal{B}}(r, \varepsilon))
\end{aligned}$$

where $\bar{\mathcal{B}}(r, \varepsilon)$ is the event complementary to $\mathcal{B}(r, \varepsilon)$. Due to (15) and (52) we have

$$\mathbf{P}(X_r^* > 0) = \mathbf{P}(T^* > r) = o(A^r) \text{ for some } A < 1. \quad (57)$$

It follows that in view of (15), (55) and (56) one can find N such that

$$\mathbf{P}(T^* > N \log r) + \mathbf{P}(\bar{\mathcal{B}}(r, \varepsilon)) = o(r^{-\kappa^*})$$

while using (2) and

$$\mathbf{P}(W_T^* > y) \sim C^* y^{-\kappa^*}, \quad C^* = C^*(\kappa^*) \in (0, \infty),$$

one can show, arguing as in Theorem 3, that for $\kappa^* < 1$

$$\begin{aligned}
&\liminf_{\varepsilon \downarrow 0} \lim_{r \rightarrow \infty} r^{\kappa^*} \mathbf{E} \left[1 - e^{W_T^* \log Q_{ra+2\varepsilon r}} \right] \\
&= \liminf_{\varepsilon \downarrow 0} \lim_{r \rightarrow \infty} \frac{r^{\kappa^*}}{-\log Q_{ra+2\varepsilon r}} \int_0^\infty \mathbf{P}(W_T^* > x) e^{x \log Q_{ra+2\varepsilon r}} dx \\
&= \liminf_{\varepsilon \downarrow 0} \Gamma(1 - \kappa^*) C^* \left(\frac{2}{m_2 a (1 + 2\varepsilon)} \right)^{\kappa^*}
\end{aligned}$$

giving for $\kappa^* < 1$

$$\liminf_{r \rightarrow \infty} r^{\kappa^*} \mathbf{P}(Z_r^* > 0) \geq K_*, \quad K_* = \Gamma(1 - \kappa^*) C^* \left(\frac{2}{m_2 a} \right)^{\kappa^*}.$$

A similar upper bound in view of (7) and (57) yields

$$\lim_{r \rightarrow \infty} r^{\kappa^*} \mathbf{P}(X_r^* + Z_r^* > 0) = K_*.$$

If $\kappa^* \geq 1$ then condition (53) entails

$$\mathbf{P}(|\tau_r - ra| > \varepsilon r) = o(r^{-1})$$

and, as before, this implies

$$\lim_{r \rightarrow \infty} (q_{\kappa^*}(r))^{-1} \mathbf{P}(X_r^* + Z_r^* > 0) = K_*,$$

where

$$K_* = \frac{2}{m_2 a} \cdot \begin{cases} C^*, & \text{if } \kappa^* = 1, \\ \int_0^\infty \mathbf{P}(W_T^* > x) dx, & \text{if } \kappa^* > 1. \end{cases}$$

We proceed by recalling (51). Now for any $\varepsilon \in (0, 1)$ we have

$$\begin{aligned} \mathbf{P}(X_{N(n)}^* + Z_{N(n)}^* > 0) &= \mathbf{P}(X_{N(n)}^* + Z_{N(n)}^* > 0; N(n) \geq a^{-1}n(1-\varepsilon)) \\ &\quad + O(\mathbf{P}(N(n) < a^{-1}n(1-\varepsilon))), \end{aligned}$$

and as $n \rightarrow \infty$

$$\begin{aligned} \mathbf{P}(X_{N(n)}^* + Z_{N(n)}^* > 0; N(n) \geq a^{-1}n(1-\varepsilon)) \\ \leq \mathbf{P}(X_{a^{-1}n(1-\varepsilon)}^* + Z_{a^{-1}n(1-\varepsilon)}^* > 0) \sim q_{\kappa^*}(a^{-1}n(1-\varepsilon)) K_*. \end{aligned}$$

It follows from [14] and our conditions that

$$\mathbf{P}(N(n) < a^{-1}n(1-\varepsilon)) = \mathbf{P}(S_{a^{-1}n(1-\varepsilon)} > n) = o(n^{-\min(\kappa^*, 1)}), \quad n \rightarrow \infty.$$

Thus,

$$\begin{aligned} \limsup_{n \rightarrow \infty} (q_{\kappa^*}(n))^{-1} \mathbf{P}(X_n + Z_n > 0) \\ \leq \lim_{\varepsilon \downarrow 0} \limsup_{n \rightarrow \infty} (q_{\kappa^*}(n))^{-1} \mathbf{P}(X_{N(n)}^* + Z_{N(n)}^* > 0) \end{aligned}$$

so that

$$\limsup_{n \rightarrow \infty} (q_{\kappa^*}(n))^{-1} \mathbf{P}(X_n + Z_n > 0) \leq K^*, \quad K^* = a^{\min(1, \kappa^*)} K_*.$$

The corresponding lower bound is obtained similarly. □

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